

Bel-Robinson energy and the nature of singularities in isotropic cosmologies

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Abstract. We review our recent work on the classification of finite time singularities that arise in isotropic universes. This scheme is based on the exploitation of the Bel Robinson energy in a cosmological setting. We comment on the relation between geodesic completeness and the Bel Robinson energy and present evidence that relates the divergence of the latter to the existence of closed trapped surfaces.

1. Introduction

In [1] we provided necessary conditions for the appearance of finite time singularities in isotropic universes based entirely on the behaviour of the Hubble parameter. These conditions provided us with a new classification of such singularities, a scheme which was further refined and expanded in [2] to include all known types of singularities and predict new ones. This second approach exploited the Bel Robinson energy, a kind of energy of the gravitational field projected in a sense to a slice in spacetime. Combining the quantities that constitute the Bel Robinson energy with the pressure and density of the matter fields through the field equations, we were thus led to a clear picture of how the matter fields influence the nature of the singularities. A complete classification was then obtained with the use of the Hubble parameter and the Bel Robinson energy.

In the next Section, we introduce the Bel-Robinson energy and show that a closed Robertson-Walker (in brief RW) universe is geodesically complete if this energy is bounded. We present a summary of our classification scheme in Section 3, and in Section 4 we give necessary and sufficient conditions for the character of the resulting singularities. In the last Section, we discuss the existence of closed trapped surfaces in terms of the Bel-Robinson energy.

2. Bel Robinson energy and completeness

Consider a sliced spacetime with metric

$$^{(n+1)}g \equiv -N^2(\theta^0)^2 + g_{ij} \theta^i \theta^j, \quad \theta^0 = dt, \quad \theta^i \equiv dx^i + \beta^i dt, \quad (1)$$

where $N = N(t, x^i)$ is the lapse function and $\beta^i(t, x^j)$ is the shift function, and the 2-covariant spatial electric and magnetic tensors

$$\begin{aligned} E_{ij} &= R_{i0j}^0, \\ D_{ij} &= \frac{1}{4} \eta_{ihk} \eta_{jlm} R^{hklm}, \\ H_{ij} &= \frac{1}{2} N^{-1} \eta_{ihk} R_{0j}^{hk}, \\ B_{ji} &= \frac{1}{2} N^{-1} \eta_{ihk} R_{0j}^{hk}, \end{aligned}$$

where η_{ijk} is the volume element of the space metric \bar{g} . These four time-dependent space tensors comprise what is called a *Bianchi field*, (E, H, D, B) , a very important frame field used to prove global in time results, cf. [3].

The *Bel-Robinson energy at time t* is given as the space integral

$$\mathcal{B}(t) = \frac{1}{2} \int_{\mathcal{M}_t} (|E|^2 + |D|^2 + |B|^2 + |H|^2) d\mu_{\bar{g}_t}, \quad (2)$$

where by $|X|^2 = g^{ij} g^{kl} X_{ik} X_{jl}$ we denote the spatial norm of the 2-covariant tensor X . In the following, we focus on a RW geometry filled with various forms of matter and with a metric given by

$$ds^2 = -dt^2 + a^2(t) d\sigma^2, \quad (3)$$

where $d\sigma^2$ denotes the usual time-independent metric on the 3-slices of constant curvature k . For this spacetime, the norms of the magnetic parts $|H|, |B|$ are identically zero while $|E|$ and $|D|$, the norms of the electric parts, reduce to

$$|E|^2 = 3(\ddot{a}/a)^2 \quad \text{and} \quad |D|^2 = 3\left((\dot{a}/a)^2 + k/a^2\right)^2. \quad (4)$$

Therefore the Bel-Robinson energy becomes

$$\mathcal{B}(t) = \frac{C}{2} (|E|^2 + |D|^2), \quad (5)$$

where C is the constant.

It is not difficult to show that in a closed RW universe such that $|D|$ is bounded above, H must be bounded above and the scale factor bounded below. Therefore H must be integrable and the spatial metric bounded below, that is such a universe is regularly hyperbolic. This in turn means that all hypotheses of the completeness theorem proved in [4] are satisfied and therefore such a universe is g -complete. It is also straightforward to see that the null energy condition is equivalent to the inequality $|E| \leq |D|$, hence completeness is then accompanied with $|E|$ being bounded above. (For a flat spatial metric we would need to impose the regular hyperbolicity hypothesis in order to conclude completeness since the latter is independent from the boundedness of $|D|$.)

3. Classification of singularities

We are interested in finding the possible types of singularities that can arise in the evolution of a RW geometry. These types result from the possible behaviours exhibited by the different triplets consisting of the scale factor a , the Hubble expansion rate H and the Bel Robinson energy \mathcal{B} . Assuming that the model has a finite time singularity at $t = t_s$, the possible behaviours of the functions in the triplet $(H, a, (|E|, |D|))$ are classified as follows:

S_1 H non-integrable on $[t_1, t]$ for every $t > t_1$

S_2 $H \rightarrow \infty$ at $t_s > t_1$

S_3 H otherwise pathological

N_1 $a \rightarrow 0$

N_2 $a \rightarrow a_s \neq 0$

N_3 $a \rightarrow \infty$

B_1 $|E| \rightarrow \infty, |D| \rightarrow \infty$

B_2 $|E| < \infty, |D| \rightarrow \infty$

B_3 $|E| \rightarrow \infty, |D| < \infty$

B_4 $|E| < \infty, |D| < \infty$.

The nature of a prescribed singularity is thus characterized completely by specifying the components in triplets of the form

$$(S_i, N_j, B_l),$$

with the indices i, j, l taking their respective values as above.

For fluid-filled RW models the various behaviours of the Bel-Robinson energy density can be written equivalently in terms of the density and pressure of the cosmological fluid:

$$B_1 \Leftrightarrow \mu \rightarrow \infty \text{ and } |\mu + 3p| \rightarrow \infty$$

$$B_2 \Leftrightarrow \mu \rightarrow \infty \text{ and } |\mu + 3p| < \infty$$

$$B_3 \Leftrightarrow \mu < \infty \text{ and } |\mu + 3p| \rightarrow \infty \Leftrightarrow \mu < \infty \text{ and } |p| \rightarrow \infty$$

$$B_4 \Leftrightarrow \mu < \infty \text{ and } |\mu + 3p| < \infty \Leftrightarrow \mu < \infty \text{ and } |p| < \infty.$$

Following [2], we may gain further insight into the nature of any of the singularities expounded above by studying the asymptotic behaviour of the three functions that determine the character of the singularity. This is done by finding the relative strength, i.e., the limit as $t \rightarrow t_s$ of the relative rate of decay, for each pair of these functions. As an example compare the standard big bang singularities in the dust and radiation models in general relativity; even though both of singularities are of the type (S_1, N_1, B_1) , they are really different since the radiation one has relative strength $a \ll H \ll (|E| \leftrightarrow |D|)$ (where ' \ll ', ' \leftrightarrow ', ' \sim ' mean that the relative limit is 0, 1 or $c \neq 0, 1$, with c a constant, respectively), whereas that in the dust models is characterized by the asymptotic behaviour $a \ll H \ll (|E| \sim |D|)$.

4. Necessary and sufficient conditions for an (S_i, N_j, B_l) type of singularity

Flat RW universes are clearly the simplest to analyze but in the long run one is interested to see whether the behaviours known to hold in the flat case continue to remain valid in more generic curved models. In this Section, we give necessary and sufficient conditions for the singularities in given flat RW models to continue to hold in more general universes in the sense of either having nonzero values of k or described by solutions in which some or all of the arbitrary constants remain arbitrary (particular or general solutions, cf. [5] for this terminology). Usually the curvature term in a typical Friedman-type equation turns out to be *subdominant* compared to the density term or, in any case, does not alter the H behaviour. Some examples of what can happen in general relativistic models are shown in the following theorems.

Theorem 1. *Necessary and sufficient conditions for an (S_2, N_3, B_1) singularity occurring at the finite future time t_s in an isotropic universe filled with a fluid with equation of state $p = w\mu$, are that $w < -1$ and $|p| \rightarrow \infty$ at t_s .*

Theorem 2. *A necessary and sufficient condition for an (S_2, N_2, B_1) singularity at t_s in an isotropic universe filled with a fluid with equation of state $p + \mu = -B\mu^\beta$, $\beta > 1$, is that $\mu \rightarrow \infty$ at t_s .*

Theorem 3. *A necessary and sufficient condition for an (S_3, N_2, B_3) singularity at t_s in an isotropic universe filled with a fluid with equation of state $p + \mu = -C(\mu_0 - \mu)^{-\gamma}$, $\gamma > 0$, is that $\mu \rightarrow \mu_0$ at t_s .*

Theorem 4. *A necessary and sufficient condition for an (S_1, N_1, B_1) singularity at t_1 in an open or flat isotropic universe filled with a fluid with equation of state $p + \mu = \gamma\mu^\lambda$, $\gamma > 0$ and $\lambda < 1$, is that $\mu \rightarrow \infty$ at t_1 .*

Theorem 5. *A necessary and sufficient condition for an (S_1, N_1, B_1) singularity at t_1 in an isotropic universe with a massless scalar field is that $\dot{\phi} \rightarrow \infty$ at t_1 .*

We can easily find examples of the behaviours described in the above theorems. The phantom cosmologies of [6] are described by Theorem (1), the various solutions of [7] can be all accommodated by theorems (2) and (3) respectively. The graduated inflationary models constructed by Barrow in [8] when $\lambda = 3/4$ belong to the situation of Theorem (4) whereas the singularities of massless scalar field model of Ref. [9] are described by Theorem (5). We shall meet the latter two models again in the next Section. The proofs of theorems (1)-(5) and the details of the dark energy examples can be found in [2], our basic reference for all these results.

One of the virtues of our classification scheme is that it can pick up even very mild singularities. The mildest in a sense type of singularity known to the authors was discovered in [7]. This is a general relativistic, flat RW universe filled with a fluid satisfying the equation of state

$$p + \mu = -\frac{AB\mu^{2\beta-1}}{A\mu^{\beta-1} + B}, \quad 0 < \beta < 1/2,$$

and for $\beta = 1/5$ it admits the exact solution $a = a_0 e^{\tau^{8/3}}$. Then $H = (8/3)\tau^{5/3}$, $\dot{H} = (40/9)\tau^{2/3}$ and $\ddot{H} = (80/27)\tau^{-1/3}$. Thus as $\tau \rightarrow 0$, a , \dot{a} , \ddot{a} , H , \dot{H} all remain finite whereas \ddot{H} becomes divergent. In this universe the Bel Robinson energy at an initial time, $\mathcal{B}(0)$, is finite whereas its time derivative is

$$\dot{\mathcal{B}}(\tau) = 3 \left[2 \frac{\ddot{a}}{a} (\ddot{H} + 2H\dot{H}) + 4 \left(\frac{k}{a^2} + H^2 \right) \left(-\frac{kH}{a^2} + H\dot{H} \right) \right] \quad (6)$$

and thus $\dot{\mathcal{B}}(\tau) \rightarrow \infty$ at $\tau \rightarrow 0$. This is like a B_4 singularity in our scheme. Since the derivative of the Bel-Robinson energy diverges, we may interpret this singularity geometrically as one *in the velocities of the Bianchi (frame) field*. At t_s the Bianchi field encounters a cusp and its velocity diverges there.

The character of the singularities can be expressed in terms of the functions describing the matter fields. We shall end this Section with a result of this sort. Note that the necessary conditions for singularities given in [1] and based on the behaviour of the Hubble parameter alone can be rephrased in terms of the electric parts of Bel Robinson energy. We have the following result concerning the nature of the spacetime singularities in such models.

Theorem 6. *Necessary conditions for null and timelike geodesically incomplete, globally and regularly hyperbolic FRW universes are that at a finite time:*

- B_1 $|E| \rightarrow \infty$ and $|D| \rightarrow \infty \Leftrightarrow \mu \rightarrow \infty$ and $|\mu + 3p| \rightarrow \infty$
 B_2 $|E| < \infty$ and $|D| \rightarrow \infty \Leftrightarrow \mu \rightarrow \infty$ and $|\mu + 3p| < \infty$
 B_3 $|E| \rightarrow \infty$ and $|D| < \infty \Leftrightarrow \mu < \infty$ and $|\mu + 3p| \rightarrow \infty \Leftrightarrow \mu < \infty$ and $|p| \rightarrow \infty$
 B_4 $|E| < \infty$ and $|D| < \infty \Leftrightarrow \mu < \infty$ and $|\mu + 3p| < \infty \Leftrightarrow \mu < \infty$ and $|p| < \infty$.

5. Incompleteness and Bel-Robinson energy

It was recently shown by Ellis in [11] that a RW space with scale factor $a(t)$ admits a past closed trapped surface if the following condition is satisfied:

$$\dot{a}(t) > \left| \frac{f'(r)}{f(r)} \right|, \quad (7)$$

with $f(r) = \sin r, r, \sinh r$ for $k = 1, 0, -1$ respectively. Recall that a closed trapped surface is a 2-surface with spherical topology such that both families of incoming and outgoing null geodesics orthogonal to the surface converge. Since our function $|D|$ can be written in the form

$$|D| = \frac{\sqrt{3}}{a^2(t)} \left| \left(\dot{a}(t) - \frac{f'(r)}{f(r)} \right) \left(\dot{a}(t) + \frac{f'(r)}{f(r)} \right) + \frac{1}{f^2(r)} \right|, \quad (8)$$

we see that the condition for the existence of a closed trapped surface becomes equivalent to the following inequality:

$$|D| > \frac{\sqrt{3}}{a^2(t)f^2(r)}. \quad (9)$$

We thus conclude that collapse singularities (as predicted by the existence of a trapped surface) are characterized by a divergent Bel Robinson energy.

Consider for example the flat graduated inflationary model of [8]. This universe satisfies the null energy condition $\mu + p = \gamma\mu^{3/4} \geq 0$, $\gamma > 0$ and the scale factor is given by

$$a(t) = \exp \left(-\frac{16}{3^{3/2}\gamma^2 t} \right), \quad (10)$$

so that the inequality

$$\dot{a}(t) = \frac{16}{3^{3/2}\gamma^2} \frac{1}{t^2} \exp \left(-\frac{16}{3^{3/2}\gamma^2 t} \right) > \frac{1}{r} \quad (11)$$

is satisfied for large r . We conclude that there exist past closed trapped surfaces in this model and by the singularity theorems we find that this model must be null geodesically incomplete. The nature of this singular behaviour is described by saying that it is of type (S_1, N_1, B_1) .

Another example is the flat RW filled with a massless scalar field given in [9]. This satisfies the null energy condition and has a past closed trapped surface: for large r we have

$$\dot{a}(t) = \frac{1}{3t^{2/3}} > \frac{1}{r}. \quad (12)$$

It is therefore null geodesically incomplete of type (S_1, N_1, B_1) .

As a last example, consider the sudden singularity first introduced in [10] which can arise in an RW model with scale factor given by

$$a(t) = \left(\frac{t}{t_s} \right)^q (a_s - 1) + 1 - \left(1 - \frac{t}{t_s} \right)^n, \quad (13)$$

where $1 < n < 2$ and $0 < q \leq 1$. We see that at t_s , $a(t) \rightarrow a_s$, $\dot{a}(t) \rightarrow \dot{a}_s$ and $\ddot{a}(t) \rightarrow -\infty$. When $\mu \geq 0$ and $p \geq 0$, we find that there exists a future closed trapped surface (the analogous inequality of (7) is $\dot{a}(t) < -|f'(r)/f(r)|$) since we have that the condition

$$\dot{a}(t) = \frac{q}{t_s^q} t^{q-1} (a_s - 1) + \frac{n}{t_s^n} (t_s - t)^{n-1} < -|\cot r| \quad (14)$$

is satisfied for $r = \pi/2$, $a_s < 1$ and $0 < t < t_s$. Therefore this model is timelike and null geodesically incomplete in the future with $|E| \rightarrow \infty$ but $|D| < \infty$ (type (S_3, N_2, B_3)).

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